Sublinear Algorithms October 8, 2015

Lecture 12

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1 Introduction

Today we will cover a sublinear time algorithm which tests whether a given image satisfies the half-plane property or ϵ-far from satisfying it. We will present a tester by Berman, Murzabulatov, and Raskhodnikova [BMR] whose query complexity is $O(1/\epsilon)$. The image is represented as a matrix of pixels, and each of the pixels is either black or white. Conceptually, an image satisfies the half-plane property if there exists a line which separates the black pixels and white pixels, i.e., pixels on one side are all black and pixels on another side are all white. Some of the following materials are copied verbatim from [BMR, BMR15] and the lecture slides with slight modifications.

2 Preliminaries and Problem Formulation

Image Representation: An image is a $n \times n$ binary matrix, where each pixel on the matrix is either black (represented as 1) or while (represented as 0).

Distance Measurements: For two matrices M_1 and M_2, the *absolute distance* $Dis(M_1, M_2)$ is the number of the pixels on which they differ; the *relative distance* $dis(M_1, M_2)$ is the absolute distance normalized by the total number of pixels, i.e., $dis(M_1, M_2) = Dis(M_1, M_2)/n^2$.

Query Access: For a matrix M, given the entry (i, j) of the pixel, the query returns the color $M(i, j)$ of the pixel.

Half-Plane Property: An image satisfies the half-plane property if there exists a line which separates the black pixels and white pixels. A image M is ϵ*-far* from the half-plane property if it needs to modify at least ϵn^2 pixels to satisfy the half-plane property.

Problem: Given an image M and a parameter ϵ, a half-plane tester should
1) accept the image with probability at least 2/3 if M satisfies the half-plane property; and
2) reject the image with probability at least 2/3 if M is ϵ-far from the half-plane property.

3 Algorithm

The algorithm is displayed in Algorithm 1.

Algorithm 1: A ϵ-half-plane tester.

input : An $n \times n$ image M; the parameter ϵ
output: *accept* or *reject*

Randomly choose $s = 18/\epsilon$ pixels uniformly and independently.
Find convex hull of black samples and convex hull of white samples.
If The two hulls intersect **then** *reject*;
otherwise, *accept*.

4 Proof Idea - Central Points

The proof idea is to identify some points that has high probability to reside in the convex hull of black points, called *black-central points*.[1] Note that the central point does not need to be a black pixel. A black-central point is defined in Definition 1. An illustration is displayed in Figure 1.

Definition 1 (black-central point)**.** *A point (i, j) is black-central if it is the intersection of two lines such that each quadrant formed by the lines has $\epsilon n^2/4$ black pixels.*

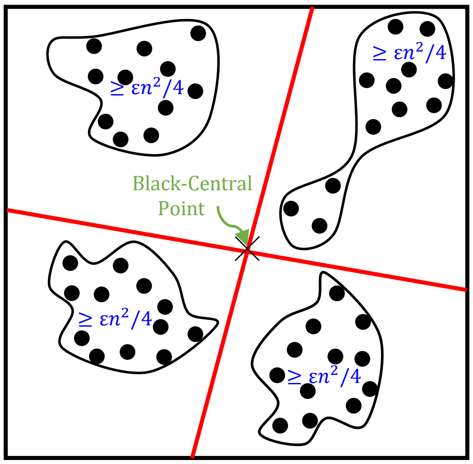

Figure 1: An illustration of a black-central point.

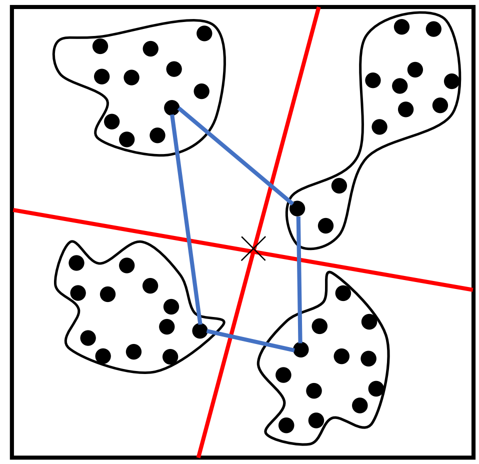

Figure 2: An illustration of a black-central point and its witness.

As Figure 2 depicts, if we can sample at least a point from the four quadrant, the black-central point will be included in the convex hull, called *witness*. We sample $s = \frac{\ln 100}{\epsilon/4} = \frac{18}{\epsilon}$ nodes to catch such witness. Consider any quadrant Q in Figure 2, and the set of all black points in Q as Q_b. Since Q_b contains at least $\epsilon n^2/4$ black pixels. Thus,

$$Pr[\text{a random point is in } Q_b] \geq \epsilon/4,$$
$$Pr[\text{a random point is not in } Q_b] < 1 - \epsilon/4,$$
$$Pr[\text{all } s \text{ random point are not in } Q_b] < (1 - \epsilon/4)^s \leq e^{-\frac{\epsilon}{4}s} = e^{-\frac{\epsilon}{4}\frac{\ln 100}{\epsilon/4}} = 1/100.$$

Since there are four quadrants, the probability that we fail to capture the witness (i.e., the s sample nodes does not have black nodes from all 4 quadrants), is less than 4/100 by union bound.

A reader may have problem about the existence of central points. Does a black-central (or white-central) point always exist? The answer is yes according to the following Ham Sandwich Theorem.

Theorem 2. *In n dimensions, any n measurable sets can be simultaneously bisected (w.r.t. their measure) by an $(n-1)$-dimensional hyperplane.*

First, we can project all nodes to y-axis (i.e., the line $x = 0$).[2] By Theorem 2 with $n = 1$, the set of all nodes can be bisected into two sets by a point, say y_b, which means that all nodes (in two-dimension space) can be bisected by the line $y = y_b$. Denote the sets after bisection as S_1 and S_2. Then by applying Theorem 2 again with $n = 2$, S_1 and S_2 can be simultaneously bisected by another line.[3] In other words, there exists two lines that can divide the nodes into four quadrants. Thus a black-central (or white-central) point must exist.

[1] Analogously, we can find the white-central points of this image.
[2] Note that the line can be chosen arbitrarily.
[3] Note that the second line must intersect with $y = y_b$ since any line parallel to $y = y_b$ can only divide either S_1 or S_2.

5 Correctness Proof

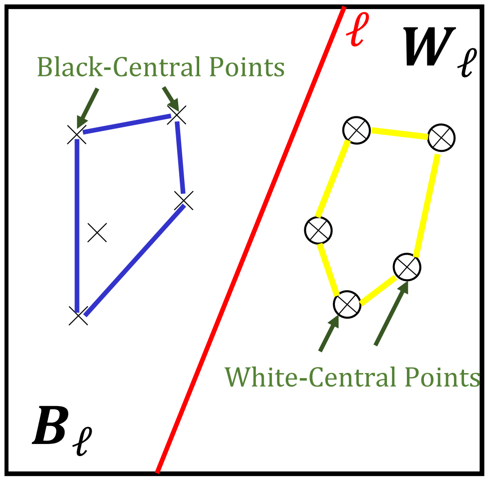

Figure 3: An illustration of non-intersecting convex hulls of black-central and white-central nodes.

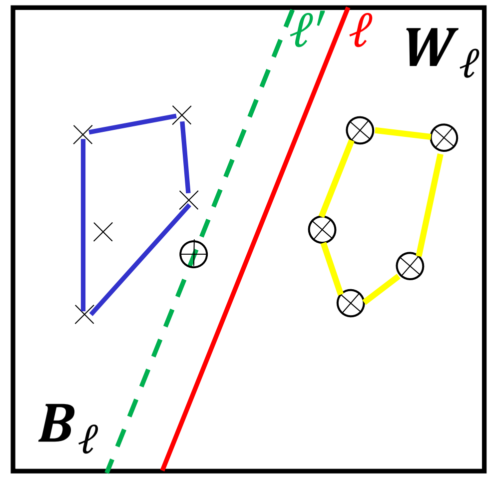

Figure 4: An illustration of how the line l' is obtained in Lemma 3.

We prove the correctness by the following lemma and theorem.

Lemma 3. *If the image is ϵ-far from satisfying the half-plane property then the convex hull of black-central nodes intersects the convex hull of white-central nodes.*

Proof. The proof is done by contradiction. First, we assume that the convex hull of black-central nodes does not intersect with the convex hull of white-central nodes. Thus, there must be a line l which separates the convex hulls, as Figure 3 shows. Let B_l and W_l be the closed half-planes formed by l, with black-central and white-central points, respectively. Thus, at least one of the following situations must hold: 1) There are $\geq \frac{\epsilon n^2}{2}$ black pixels in W_l or 2) There are $\geq \frac{\epsilon n^2}{2}$ white pixels in B_l.[4] Suppose that (2) happens. Let l' be the line parallel to l and furthest from l such that there are $\frac{\epsilon n^2}{2}$ white pixels in closed half-plane to the left of l', as Figure 4 shows. Also, there are $\frac{\epsilon n^2}{2}$ white pixels in closed half-plane to the right of l'. Thus, by Ham Sandwich Theorem, there is another line that can bisect the white pixels on 1) the white pixels in the closed half-plane to the left of l' and 2) the white pixels in the closed half-plane to the right of of l'. In other words, there is a white-central points on l'. There is a contradiction and the lemma follows. □

According to Lemma 3, the convex hulls of black-central nodes and white-central nodes intersect if the image is ϵ-far from satisfying the half-plane property, i.e., some points (denoted as v) are in the both convex hulls. Moreover, v is in 1) the convex hull of at most three black-central nodes and 2) the convex hull of at most three white-central nodes, as Figure 5 shows. Recall that we fail to capture a central point with probability $< 4/100$ (as illustrated in Section 4. Thus, by union bound, we fail to capture one or multiple of the 6 central points with probability $< 24/100 < 1/3$.

Thus, we can obtain the correct theorem.

Theorem 4. *Algorithm 1 is a ϵ-tester of half-plane property*

Proof. If an image M satisfies the half-plane property, Algorithm 1 always accepts since the convex hull of all black pixels does not intersect with the convex of all white pixels. If an image M is ϵ-far from satisfying the

[4] Otherwise, the image is ϵ-close to be a half-plane since we can change all black pixels in W_l to be white and all white pixels in B_l to be black to become half-plane, and the total number of changed pixels is $< \epsilon n^2$).

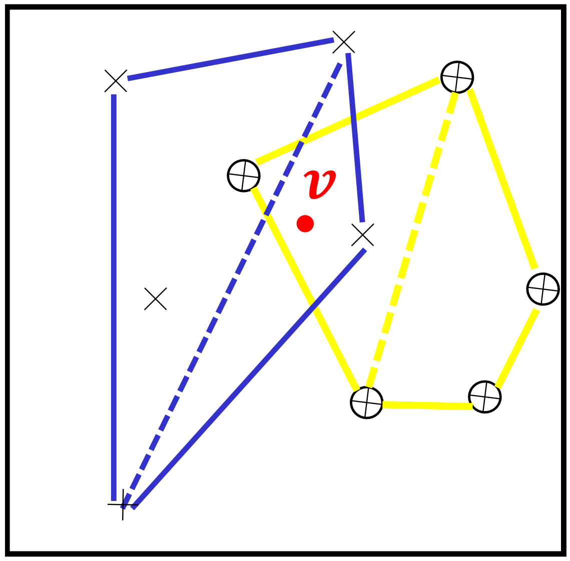

Figure 5: An illustration for node v in the intersection of black-central nodes and white-central nodes

half-plane property, Algorithm 1 rejects M with probability $> 2/3$ according to Lemma 3 and the following descriptions. The theorem follows. □

References

[BMR] Piotr Berman, Meiram Murzabulatov, and Sofya Raskhodnikova. The role of adaptivity in image testing. *Manuscript*.

[BMR15] Piotr Berman, Meiram Murzabulatov, and Sofya Raskhodnikova. Constant-time testing and learning of image properties. *CoRR*, abs/1503.01363, 2015.